Numerical Approximation of Spectra for Localized Oscillatory Structures

Nadejda Drenska, thesis advisor: Björn Sandstede

Division of Applied Mathematics, Brown University, RI 02912, USA
e-mail: nadejda_drenska@brown.edu, bjorn_sandstede@brown.edu

Introduction

Systems of time-periodic differential equations arise in various models drawn from biology, chemistry and physics. One example, and the motivation of the current research, is the investigation of stability of oscillons: radially symmetric solutions, which are localized in space and periodic in time. Oscillons were first observed in experiments in 1996 when granular materials and clay suspension were vibrated vertically (figure 1). Different amplitudes and frequencies result in different pattern formations. Reaction-diffusion systems, such as models of light-sensitive Belousov-Zhabotinsky reactions, possess the same dynamics.

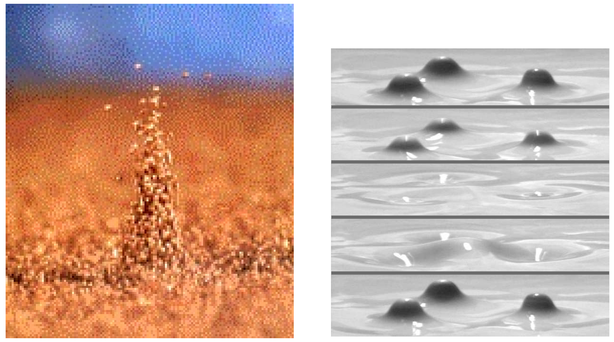

Figure 1: Oscillons in vibrated granular materials [1] [left] and in vibrated clay suspension [2] [right]

There are two traditional methods for finding oscillons in mathematical models and determining their stability: initial value problem solvers and continuation (using Newton's method). The continuation method is computationally inexpensive and more systematic, relative to the initial value problem solvers. Unlike initial value problem solvers, the continuation method finds both stable and unstable oscillons, but it requires a way to determine their stability (that is, whether they disappear over time). A novel method for determining stability of both Partial Differential Equations (PDEs) and Ordinary Differential Equations (ODEs) is presented in the current paper. For ODEs, the idea relies on the fact that if the absolute values of all the eigenvalues of the flow (so-called Floquet multipliers) are at most one, then the oscillon is stable. Hence an accurate approximation of the spectrum (or just of the important eigenvalues - the biggest ones) is sufficient to determine whether the oscillon is stable. For PDEs, we can use the same techniques if we first discretize over time and space, and then apply the theory to the resulting systems of ODEs. In the current thesis we have assumed that a time-periodic system of ODEs is given, and will not consider whether it originates from a PDE model. The paper presents the theory behind a numerical algorithm of finding the spectrum of a given time-periodic system of ODEs. Instead of calculating the full solution map of the ODE and calculating its spectrum, we pose the problem as an eigenvalue problem for an appropriate operator. The problem is then reformulated in a more convenient way by breaking down this operator into an invertible and a non-invertible part, so that the spaces considered are essentially $\mathbb{R}^N$ and we have standard perturbation theory available to us. A key Rouche-type theorem states that, under appropriate conditions, if two operators are close, then their spectra are close, that is, a good approximation is obtained. The two operators considered are the original one and its Galerkin approximation to N Fourier modes, which, if chosen properly, represents the truncation performed when the algorithm is implemented on a computer.

The Problem

Find all λ for which

$$(\frac{\partial}{\partial t} + \mathrm{I} - A(t) - \lambda)u := (L - \lambda)u = 0 \tag{1}$$

has a solution $u \neq 0$.

Reformulation of the Problem

There are two main difficulties which drive the reformulation of the problem into one which can be considered numerically. The operator L that we consider is originally posed as $L : H^1_{per}(0,p) \to L^2_{per}(0,p)$. A possible way of investigation would be to project it to $\mathbb{R}^N : L_N : \mathbb{R}^N \to \mathbb{R}^N$ and to investigate how the spectrum of L_N compares to that of L. However, it would be difficult to compare spaces $\mathbb{R}^N$ and $H^1_{per}(0,p)$, especially when we want the dimension N to increase. At the same time, $L : H^1_{per}(0,p) \to L^2_{per}(0,p)$ is awkward as $L : L^2_{per}(0,p) \to L^2_{per}(0,p)$ is not bounded and it is difficult to apply perturbation theory directly. Hence, we reformulate the problem as follows: we split L as $L = L_0 - A(t)$ where $L_0 = \frac{\partial}{\partial t} + \mathrm{I}$. Note that $L^0 : H^1_{per}(0,p) \to L^2_{per}(0,p)$ is invertible and has a bounded inverse that is also compact. Then, the problem to find all λ for which

$$(L - \lambda)u = 0 \tag{2}$$

has a solution $u \neq 0$, is equivalent to solving

Results In Pictures

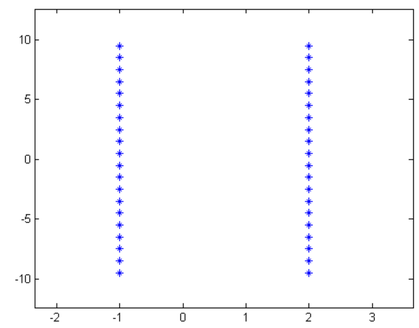

Figure 2: Floquet exponents lie on lines

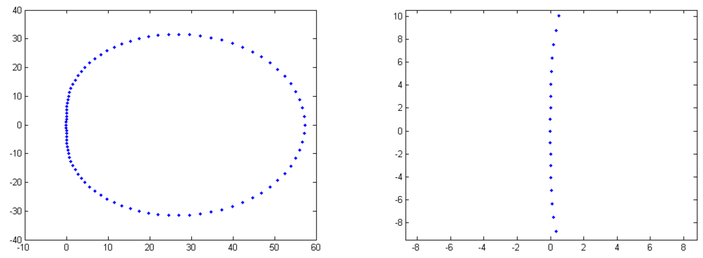

Figure 3: Backward differences approximation of order 1 (left) with $v' = 0v$ on $[0, 2\pi]$ with $m = 80$ mesh points. A zoomed-in image near the origin (right) shows that the approximation near the origin is good.

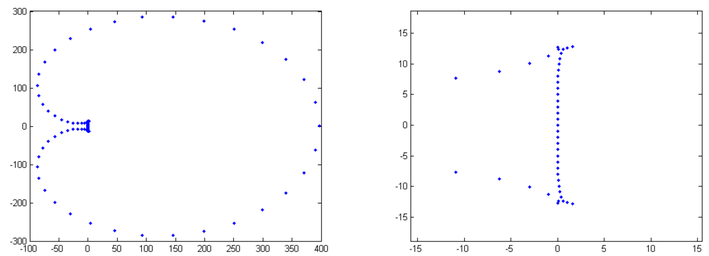

Figure 4: Backward differences approximation of order 6 (left) $v' = 0v$ on $[0, 2\pi]$ with $m = 80$ mesh points. A zoomed-in image near the origin (right) shows that the approximation near the origin is good, in fact a lot better than the backward difference approximation of order 1.

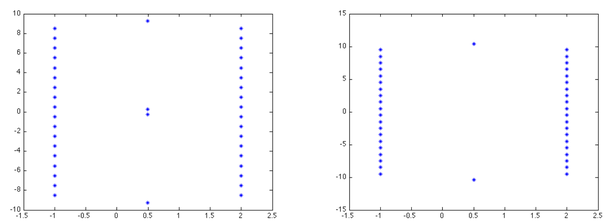

Figure 5: Spectra using the Galerkin method with 20 modes (left) and 21 modes (right). Spurious eigenvalues (near 0.5) occur when the number of modes is even.

$$Fu := L_0^{-1}(L - \lambda)u = (\mathrm{I} - L_0^{-1}(A + \lambda))u = 0 \tag{3}$$

The truncation T_N we will consider is

$$T_N u := (\mathrm{I} - Q_N L_0^{-1}(A + \lambda))u = 0, \tag{4}$$

where $Q_N : L^2_{per}(0,p) \to L^2_{per}(0,p)$ projects onto a $(2N+1)$-dimensional subspace. We analyze the problem in this setup, using standard perturbation theory. The outline of the paper is the following. First, we compare the analytical formulation and the practical implementation and prove equivalence under certain conditions. Then, we show that the difference between the truncated and the full operator

$$W_N := T_N - F = (\mathrm{I} - Q_N)L_0^{-1}(A + \lambda)$$

is small in norm. Next, we explain what we mean by eigenvalue multiplicity, so that we can prove that the eigenvalues of the full and of the truncated operator are close, too. Finally, we present in more detail a comparison between finite differences methods and the method developed in the paper.

Theoretical Results

Theorem. *(No Spurious Eigenvalues and Multiplicity Conservation) For every arbitrarily small $\epsilon > 0$ there exists a natural number N_0 such that for each $N > N_0$ all the eigenvalues of the approximation*

$$T_N = [\mathrm{I} - Q_N L_0^{-1}(A + \lambda)]v = 0$$

belong to the union of ϵ-balls around the Floquet exponents. Moreover, their multiplicity is preserved: if the ϵ-balls around different Floquet exponents do not intersect each other, then there are as many eigenvalues of the approximation in each ϵ-ball as the multiplicity of that Floquet exponent.

Conclusion

In the paper we proved the uniform convergence in location and multiplicity of the numerically computed eigenvalues to the truncated operator T_N to the Floquet exponents on any bounded domain. There have been no restrictions imposed on the time-periodic matrix $A(t)$. Less general results have been shown in [4] and [5].
Deconinck and Kutz [4] work in a slightly **less** general setup, do not prove convergence of their method and have more complicated computations.
Johnson and Zumbrun [5] prove convergence in location and multiplicity **only** for operators of non degenerate type under certain conditions. Their proof uses more complicated methods than used here.

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